Combination, Permutation, Probability (2)

1. Find the number of permutations that can be formed from the letters of the word POPULAR. How many of these permutations:

(a) begin and end with P?

(b) have the two P's separated?

(c) have the vowels together?

(a) Since the P's are fixed, the other 5 letters can be permutated.

Number of permutations $= 5! = \mathbf{120}$

(b) If the two P's must be placed together, let this two P's are joined as one letter (PP), so the number of permutations $= (7-1)! = 720$

Total permutations with the two P's joined or not joined together $= \frac{7!}{2!} = 2520$

Number of permutations with the two P's separated $= 2520 - 720 = \mathbf{1800}$

(c) There are 3 vowels O,U,A , let them joined together as one letter (OUA), so there are 5 letters

$\{(OUA), P, P, L, R\}$, number of permutations $= \frac{5!}{2!} = 60$

However the vowels O,U,A can be permutated and the number of permutations $= 3! = 6$

So the number of permutations $= 60 \times 6 = \mathbf{360}$

2. A box contains 5 green marbles, 4 blue marbles and 6 red marbles. A marble is picked at random. Without replacing the first marble, another marble is taken from the box. Calculate the probability that

(a) the first marble is green and the second marble red.

(b) two marbles are NOT of the same colour.

(a) $P(G_1) = \frac{5}{5+4+6} = \frac{5}{15} = \frac{1}{3}$, $P(R_2|G_1) = \frac{6}{4+4+6} = \frac{6}{14} = \frac{3}{7}$

$P(G_1 \text{ and } R_2) = P(G_1)\ P(R_2|G_1) = \frac{1}{3} \times \frac{3}{7} = \mathbf{\frac{1}{7}}$

(b) P(two marbles are not the same colour) $= P(G_1)\ P(G_2|G_1) + P(B_1)\ P(B_2|B_1) + P(R_1)\ P(R_2|R_1)$

$= \frac{5}{5+4+6} \times \frac{4}{4+4+6} + \frac{4}{5+4+6} \times \frac{3}{5+3+6} + \frac{6}{5+4+6} \times \frac{5}{5+4+5} = \frac{31}{105}$

P(two marbles are not the same colour) $= 1 -$ P(two marbles are not the same colour)

$= 1 - \frac{31}{105} = \mathbf{\frac{74}{105}}$

3. Events A and B are events such that $P(A) = \frac{1}{2}, P(B) = \frac{1}{4}$ and $P(A \text{ or } B \text{ but not both}) = \frac{1}{3}$.

(a) Find **(i)** $P(A \cap B)$ **(ii)** $P(A' \cap B)$ **(iii)** $P(A|B)$ **(iv)** $P(B|A')$

(b) State, with explanations, whether A and B are

(i) independent **(ii)** mutually exclusive.

(a) Put $P(A \Delta B) = P(A \text{ or } B \text{ but not both}) = \frac{1}{3}$

$$P(A \cap B) = \frac{1}{2}[P(A) + P(B) - P(A \Delta B)] = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{1}{3}\right] = \mathbf{\frac{5}{24}}$$

$$P(A' \cap B) = P(B) - P(A \cap B) = \frac{1}{4} - \frac{5}{24} = \mathbf{\frac{1}{24}}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{5}{24}}{\frac{1}{4}} = \mathbf{\frac{5}{6}}$$

$$P(B|A') = \frac{P(A' \cap B)}{P(A')} = \frac{P(A' \cap B)}{1 - P(A)} = \frac{\frac{1}{24}}{1 - \frac{1}{2}} = \mathbf{\frac{1}{12}}$$

(b) $P(A \cap B) = \frac{5}{24}$, $P(A)P(B) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

$P(A \cap B) \neq P(A)P(B)$, A and B are not independent.

$P(A \cap B) = \frac{5}{24} \neq 0$, A and B are not mutually exclusive.

4. A boy plays a computer game in which he has to drive successfully a vehicle through a terrain within a certain time. In the first attempt the probability he can succeed is 0.75 and in subsequent attempts if he is successful the difficulty increases and the probability of success is half the probability of his previous attempt. If he is unsuccessful the probability remains the same. He plays three games.

(a) Find the probability he is successful in all three games.

(b) Given that he is successful in the first game, find the probability he is successful in exactly two games.

(a) the probability he is successful in all three games $= \frac{3}{4} \times \frac{3}{8} \times \frac{3}{16} = \mathbf{\frac{27}{512}}$

(b) Since he is successful in the first game, he must be successful in exactly one game in the remaining two games.

Probability = P(successful in second game and unsuccessful in third game)
+P(unsuccessful in second game and successful in third game)

$$= \frac{3}{8} \times \left(1 - \frac{3}{16}\right) + \left(1 - \frac{3}{8}\right) \times \frac{3}{8} = \mathbf{\frac{69}{128}}$$

5. Three red cards are labeled A, B and C, and seven blue cards are labeled 1, 2, 3, 4, 5, 6 and 7. Find the number of ways we can

(a) arrange all the cards in a straight line as that the cards of the same colour are next to each other,

(b) choose and arrange equal number of red and blue cards in a straight line so that the cards of the same colour are next to each other.

(a) Number of arrangement = $(3!\,7!) \times 2 = \mathbf{60480}$

(multiply by 2 since red before blue or blue before red)

(b) Number of arrangement

$= C(3,1)C(7,1) \times 2 + [C(3,2) \times 2!] \times [C(7,2) \times 2!] \times 2 + [C(3,3) \times 3!] \times [C(7,3) \times 3!] \times 2$

$= \mathbf{3066}$

6. **(a)** Given that digits 2, 3, 4, 5, 6, three digits numbers with values greater than 400 are formed. In how many ways can this be done if the numbers formed are odd and if the digits formed may not be used repeatedly.

(b) The digits of the numbers 421265 are arranged so that the resulting number is even. Find the number of ways in which this can be done.

(a) If the unit digit is 3, the hundred digit can have 3 choice, that is, 4, 5, 6 and the tenth digit can be one of the remaining 3 choice. No of ways = $n_1 = 1 \times 3 \times 3 = 9$.

If the unit digit is 5, the hundred digit can have 2 choice, that is, 4, 6 and the tenth digit can be one of the remaining 3 choice. No of ways = $n_2 = 1 \times 2 \times 3 = 6$.

Total number of ways = $9 + 6 = 15$.

(b) There are two 2's, then we have four even numbers 4,2,2,6 and there are 4 choices for the unit digit. The tenth digits have 5 remaining choices, hundred digit have remaining 4 choices,...

Since there are two 2's, we have to divide by 2 in counting the number of ways.

Therefore the number of ways = $\frac{1\times2\times3\times4\times5\times4}{2} = 240$

7. Ada and Bill are shooting at a target. The probability that Ada's shot hits the target is $\frac{1}{2}$ and the probability that Bill's shot misses the target is $\frac{1}{3}$. What is the probability that:

(a) both their shots hit the target?

(b) only one of their shots hits the target?

(c) none of their shots hits the target?

(a) $P(A) = \frac{1}{2},\ P(A') = 1 - \frac{1}{2} = \frac{1}{2}\ ,\ P(B') = \frac{1}{3},\ P(B) = 1 - \frac{1}{3} = \frac{2}{3}$

The probability that both their shots hit the target = $P(A \cap B) = P(A)P(B) = \frac{1}{2} \times \frac{2}{3} = \mathbf{\frac{1}{3}}$

(b) The probability that only one of their shots hits the target

$= P(A)P(B') + P(A')P(B) = \frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{2}{3}=\frac{1}{2}$

(c) The probability that none of their shots hits the target

$= P(A')P(B') = \frac{1}{2}\times\frac{1}{3} = \mathbf{\frac{1}{6}}$

8. The events A and B are such that $P(B) = 0.4, P(A'|B) = 0.8$ and $P(B|A') = 0.5$, find

(a) $P(A \cap B)$ **(b)** $P(A \cup B)$

(a) $P(A' \cap B) = P(A'|B)P(B) = 0.8 \times 0.4 = 0.32$

Since $(A' \cap B) \cap (A \cap B) = \emptyset$ and $(A' \cap B) \cup (A \cap B) = B$

Hence $P(A' \cap B) + P(A \cap B) = P(B) \Longrightarrow P(A \cap B) = 0.4 - 0.32 = \mathbf{0.08}$

(b) $P(A') = \frac{P(A' \cap B)}{P(B|A')} = \frac{0.32}{0.5} = 0.64$

$P(A) = 1 - P(A') = 1 - 0.64 = 0.36$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.36 + 0.4 - 0.08 = \mathbf{0.68}$

9. A and B play 8 games of tennis, 4 on Saturday and 4 on Sunday.

(a) Find the number of ways A can win 2 games on Saturday and 2 on Sunday.

(b) Find the number of ways A can win a total of 4 games over 2 days.

(c) Show that $\binom{8}{4} = \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$

(a) Number of ways A can win = $\binom{4}{2}^2$

(b) Number of ways A can win = $\binom{4}{0}\binom{4}{4} + \binom{4}{1}\binom{4}{3} + \binom{4}{2}\binom{4}{2} + \binom{4}{3}\binom{4}{1} + \binom{4}{4}\binom{4}{0}$

$= \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$ since $\binom{4}{k} = \binom{4}{4-k}$

(c) **Method 1**

A can win a total of 4 games over 2 days is the same as A can win 4 games in 8 games.

Number of ways A can win = $\binom{8}{4}$

But by (b), Number of ways A can win $= \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$

Therefore $\binom{8}{4} = \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$

Method 2

Consider $(1+x)^8 = (1+x)^4(1+x)^4$

$\binom{8}{0} + \binom{8}{1}x + \cdots + \binom{8}{4}x^4 + \cdots + \binom{8}{8}x^8$

$= \left[\binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4\right]\left[\binom{4}{4} + \binom{4}{3}x + \binom{4}{2}x^2 + \binom{4}{1}x^3 + \binom{4}{0}x^4\right]$

Compare coefficient of x^4 term, $\binom{8}{4} = \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$.

10. A boy wishes to buy exactly six marbles. There are four different colours of marbles available. In how many ways can he buy the six marbles. (Hint: Arrange 111000000)

Let the four colours be A, B, C and D.
The 1's in the number 111000000 is the separation lines.
000100101 means we choose 3 colour A, 2 colour B and 1 colour C. There is no colour D marbles.
111000000 means we choose 6 colour D marbles.
The problem then transforms to find the number of arrangements of the number 111000000.

The number of arrangements $= \frac{9!}{3!6!} = \mathbf{84}$

This problem is related to combination with repetitions, to choose k objects from n objects with repeititions, the number of repeated combinations = $\left(\binom{\boldsymbol{n}}{\boldsymbol{k}}\right) = \binom{\boldsymbol{n}+\boldsymbol{k}-\mathbf{1}}{\boldsymbol{k}} = \frac{(\boldsymbol{n}+\boldsymbol{k}-\mathbf{1})!}{\boldsymbol{k}!(\boldsymbol{n}-\mathbf{1})!}$

In this problem, the number of arrangements $= \left(\binom{4}{6}\right) = \binom{4+6-1}{6} = \binom{9}{6} = \frac{9!}{3!6!} = \mathbf{84}$

11. Ten items were taken from a large population where the probability of an item is defective 0.02.

(a) What is the probability that no defective items are taken?

(b) What is the probability that at most one item is defective?

(a) The required probability $= \binom{10}{0}(0.02)^0(0.98)^{10}$

$= (0.98)^{10} \approx \mathbf{0.81707280688754689024}$

(b) The required probability $= \binom{10}{0}(0.02)^0(0.98)^{10} + \binom{10}{1}(0.02)^1(0.98)^9$

$= (0.98)^{10} + 10 \times (0.02)(0.98)^9 \approx \mathbf{0.98382235931357686784}$

12. Experience shows that 40% of the throws of a basketballer result in ender the ring.

(a) Find the probability, out of ten throws made by the basketballer, at least two throws result in entering the ring.

(b) Find the probability that at most five throws need to be made by the basketballer so that four throws result in entering the ring.

(a) $X \sim B(10,0.4)$ (Binominal distribution)

$P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$= 1 - \binom{10}{0}(0.4)^0(0.6)^{10} - \binom{10}{1}(0.4)^1(0.6)^9$

$\approx 1 - 0.0060466176 - 0.040310784 \approx \mathbf{0.9536425984}$

(b) $X \sim B(4,0.4), Y \sim B(5,0.4)$

Required probability $= P(X=4) + [P(Y=4) - P(\text{4 good throws in the first four throws})]$

$= \binom{4}{4}(0.4)^4(0.6)^0 + \binom{5}{4}(0.4)^4(0.6)^1 - (0.4)^4(0.6)^1$

$= 0.0256 + 0.0768 - 0.01536 = \mathbf{0.08704}$

13. A type of seed is sold in packets which contain ten seeds each. On the average, it is found that a seed per packet does not germinate. Find the probability that a packet chosen at random contains less than two seeds which does not germinate.

$X \sim B(10, 0.1)$

$P(X < 2) = P(X = 0) + P(X = 1) = \binom{10}{0}(0.1)^0(0.9)^{10} + \binom{10}{1}(0.1)^1(0.9)^9$

$\approx 0.3486784401 + 0.387420489 \approx 0.7360989291$

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